Announcements

1) Exam 2 on Thursday

Directional Derivatives

The directional derivative of $z = f(x,y)$ at the point (a,b) in the direction of $v = \langle v_1, v_2 \rangle \neq \langle 0,0 \rangle$ is

$$D_v f(a,b)$$

$$= \boxed{\nabla f(a,b) \cdot \frac{v}{\|v\|}}$$

Example 1: Let

$$f(x,y) = \ln((xy)^x),$$

$(a,b) = (1,e)$,

$v = \langle 6, 2 \rangle$. Find

$D_v f(1,e)$.

$$
\begin{aligned}
f(x,y) &= \ln((xy)^x) \\
&= x\ln(xy) \\
&= x(\ln(x) + \ln(y)) \\
&= x\ln(x) + x\ln(y)
\end{aligned}
$$

$$f(x,y) = x\ln(x) + x\ln(y)$$

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

$$= \left\langle \ln(x) + 1 + \ln(y), \frac{x}{y} \right\rangle$$

$$= \left\langle \ln(xy) + 1, \frac{x}{y} \right\rangle$$

$$\nabla f(1,e) = \left\langle \ln(e) + 1, \frac{1}{e} \right\rangle$$

$$= \left\langle 2, \frac{1}{e} \right\rangle$$

$$v = \langle 6, 2 \rangle, \quad \|v\| = \sqrt{36+4}$$
$$= \sqrt{40}$$

$$D_v f(1, e)$$
$$= \nabla f(1, e) \cdot \frac{v}{\|v\|}$$
$$= \left\langle 2, \frac{1}{e} \right\rangle \cdot \left\langle \frac{6}{\sqrt{40}}, \frac{2}{\sqrt{40}} \right\rangle$$
$$= \boxed{\frac{1}{\sqrt{40}}\left(12 + \frac{2}{e}\right)}$$

The Gradient and Maximum Increase

Q: In which direction is the directional derivative increasing the fastest?

A: The direction of the gradient!

$$D_v f(a,b)$$

$$= \nabla f(a,b) \cdot \frac{v}{\|v\|}$$

$$= \|\nabla f(a,b)\| \underbrace{\left\| \frac{v}{\|v\|} \right\|}_{=1} \cos(\theta)$$

$$= \|\nabla f(a,b)\| \cos(\theta)$$

Biggest when $\cos(\theta) = 1$,

that is when $\theta = 0$, i.e.

v is parallel to $\nabla f(a,b)$.

In this case, the magnitude of the change is

$$\boxed{\left| D_{\nabla f} f(a,b) \right| = \| \nabla f(a,b) \|}$$

Example 2: Let

$$g(x,y) = (xy)^{\arctan\left(\frac{x}{y}\right)}.$$

Find the direction of maximum increase at the point $(1,1)$.

Answer $= \nabla g(1,1)$

Calculate!

$$(xy)^{\arctan(\frac{x}{y})}$$

$$= e^{\ln((xy)^{\arctan(\frac{x}{y})})}$$

$$= e^{\arctan(\frac{x}{y})\ln(xy)}$$

$$\frac{\partial g}{\partial x} = e^{\arctan(\frac{x}{y})\ln(xy)}$$

$$\frac{\partial}{\partial x}\left(\arctan(\frac{x}{y})\ln(xy)\right)$$

$$= (xy)^{\arctan(\frac{x}{y})} \cdot \left(\arctan(\frac{x}{y}) \cdot\right.$$

$$= (xy)^{\arctan\left(\frac{x}{y}\right)} \cdot$$

$$\left(\arctan\left(\frac{x}{y}\right)\frac{1}{x} + \ln(xy)\frac{1}{y\left(1+\left(\frac{x}{y}\right)^2\right)}\right)$$

evaluate at $(1,1)$ $(\ln(1)=0)$

We get

$$1^{\arctan(1)} \cdot \arctan(1) \cdot 1$$

$$= \boxed{\frac{\pi}{4}}$$

$$\frac{\partial}{\partial y}\left(e^{\ln(xy)\arctan(\frac{x}{y})}\right)$$

$$= (xy)^{\arctan(\frac{x}{y})} \cdot$$

$$\left(\arctan\left(\frac{x}{y}\right)\cdot\frac{1}{y} + \ln(xy)\frac{-x}{y^2\left(1+\left(\frac{x}{y}\right)^2\right)}\right)$$

plug in $(1,1)$ $(\ln(1)=0)$

We get

$$\boxed{\frac{\pi}{4}}$$

$$\nabla g(1,1) = \boxed{\left\langle \pi/4, \pi/4 \right\rangle}$$

Maximum rate of change

$$= \| \nabla g(1,1) \|$$

$$= \boxed{\frac{\pi}{4}\sqrt{2}}$$